Further mathematics
Higher level
Paper 1

Thursday 23 May 2019 (afternoon)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

2219 – 7101

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 12]

The graph G with vertices $\text{A}, \text{B}, \text{C}, \text{D}$ and E has the weights shown in the following table.

	A	B	C	D	E
A	–	8	11	17	12
B	8	–	14	9	13
C	11	14	–	16	10
D	17	9	16	–	15
E	12	13	10	15	–

(a) Justifying your answer, explain whether or not G contains an Eulerian circuit. [2]

(b) Prove that G cannot be drawn as a planar graph. [3]

(c) Starting at A, use the nearest-neighbour algorithm to find an upper bound for the travelling salesman problem for G. [3]

(d) By deleting vertex A, use the deleted vertex algorithm to find a lower bound for this travelling salesman problem. [4]

2. [Maximum mark: 9]

The function f is defined for $x \geq 0$ by $f(x) = \ln(2\text{e}^x - 1)$.

(a) Determine the Maclaurin series for $f(x)$ as far as the term in x^3. [7]

(b) Hence determine the value of

$$\lim_{x \to 0} \frac{f(x) - 2x}{x^2}.$$ [2]

3. [Maximum mark: 12]

(a) The matrix $\boldsymbol{A}$ is given by $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(i) Show that the eigenvalues of $\boldsymbol{A}$ are real if $(a-d)^2 + 4bc \geq 0$.

(ii) Deduce that the eigenvalues are real if $\boldsymbol{A}$ is symmetric. [6]

(b) The matrix $\boldsymbol{B}$ is given by $\boldsymbol{B} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

(i) Determine the eigenvalues of $\boldsymbol{B}$.

(ii) Determine the corresponding eigenvectors. [6]

4. [Maximum mark: 8]

The positive integer N is given by 1321 when expressed in base b and 521 when expressed in base $b+2$.

(a) Determine the value of b. [4]

(b) Express N

(i) in base 10;

(ii) in base 16. [4]

5. [Maximum mark: 9]

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\tan x = \sin x \quad \text{where } 0 \leq x < \frac{\pi}{2}$$

Given that $y = 2$ when $x = 0$, solve the differential equation giving your answer in the form $y = f(x)$. [9]

Turn over

6. [Maximum mark: 8]

The group $\{G, *\}$ has the following Cayley table.

$*$	0	1	2	3	4	5
0	4	5	0	1	2	3
1	5	2	1	4	3	0
2	0	1	2	3	4	5
3	1	4	3	0	5	2
4	2	3	4	5	0	1
5	3	0	5	2	1	4

(a) Determine the order of each of the elements of $\{G, *\}$. [4]

(b) Hence find the subgroup S_2 of order 2 and the subgroup S_3 of order 3. [2]

(c) Write down the coset with respect to S_2 of each element of $\{G, *\}$ not included in S_2. [2]

7. [Maximum mark: 12]

(a) Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt{n}}$. [5]

(b) (i) Use l'Hôpital's rule to determine the value of $\lim_{x \to 0} \frac{\ln(1+x)}{x}$.

(ii) Use the limit comparison test together with an appropriate series to determine whether the series $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$ is convergent or divergent. [7]

8. [Maximum mark: 7]

The line AD is a median of the acute-angled triangle ABC and E is the midpoint of AD.
The line BE meets AC at the point F.

(a) Draw a diagram to illustrate this situation. [1]

(b) Determine the value of the ratio $\frac{\text{CF}}{\text{AF}}$. [4]

(c) The line CE meets AB at the point G. Giving a reason, write down the value of the ratio $\frac{\text{BG}}{\text{AG}}$. [2]

9. [Maximum mark: 12]

Consider the system of equations

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ 4 & 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ \mu \end{bmatrix}$$

where the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ 4 & 1 & 6 \end{bmatrix}$ is singular and μ is a constant.

(a) Determine the value of μ for which the equations are consistent. [4]

(b) For this value of μ

(i) solve the system of equations;

(ii) find the values of x, y and z which minimize $x^2 + y^2 + z^2$ and interpret your result geometrically. [8]

10. [Maximum mark: 10]

The continuous random variable X has cumulative distribution function F given by

$$F(x) = \begin{cases} 0, \ x < 0 \\ \dfrac{2}{\pi}\arctan x, \ 0 \le x < \infty \end{cases}.$$

(a) (i) Sketch the graph of $F(x)$ for $x \ge 0$.

(ii) Explain how it can be deduced from the graph of $F(x)$ that the mode of X is zero.

(iii) Determine the median of X. [5]

(b) It is often stated that for certain probability distributions, the following approximation is true:

$$\text{Median} - \text{Mode} \approx 2(\text{Mean} - \text{Median}).$$

Explain why this approximation is not valid for the probability distribution defined above. [5]

11. [Maximum mark: 10]

(a) (i) Show that the set S_1 of three-dimensional vectors given by

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix} \right\}$$

is a basis for three-dimensional vectors.

(ii) Express the vector $\begin{bmatrix} 9 \\ 17 \\ 3 \end{bmatrix}$ in terms of S_1. [5]

(b) (i) Show that the set S_2 of three-dimensional vectors given by

$$S_2 = \left\{ \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 3 \end{bmatrix} \right\}$$

is not a basis for three-dimensional vectors.

(ii) State the dimension of the subspace spanned by S_2.

(iii) Determine whether or not the vector $\begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix}$ belongs to this subspace. [5]

12. [Maximum mark: 9]

The relation R is defined on $\mathbb{Z}^+$ such that xRy if and only if $x^2 - y^2 \equiv 0 \pmod{N}$ where $N \geq 3$ is a positive integer.

(a) Show that R is an equivalence relation for all values of N. [6]

(b) Show that $N - 1$ and $N + 1$ are in the same equivalence class as 1. [3]

13. [Maximum mark: 11]

The function $f : \boldsymbol{M} \to \boldsymbol{M}$ where $\boldsymbol{M}$ is the set of 2×2 matrices, is given by $f(\boldsymbol{X}) = \boldsymbol{AX}$ where $\boldsymbol{A}$ is a 2×2 matrix.

(a) Given that $\boldsymbol{A}$ is non-singular, prove that f is a bijection. [7]

It is now given that $\boldsymbol{A}$ is singular.

(b) By considering appropriate determinants, prove that f is not a bijection. [4]

14. [Maximum mark: 12]

The Poisson random variable X with mean m has probability function

$$P(X = x) = \frac{\mathrm{e}^{-m} m^x}{x!}, \; x \in \mathbb{N}.$$

(a) Show that the probability generating function of X is given by

$$G_x(t) = \mathrm{e}^{m(t-1)}.$$ [3]

(b) A random sample X_1, X_2, X_3 is taken from the distribution of X. The random variable Y is defined by $Y = X_1 + 2X_2 + 3X_3$.

(i) Show that the probability generating function of Y is given by $G_y(t) = \mathrm{e}^{-3m}\mathrm{e}^{m(t+t^2+t^3)}$.

(ii) By considering the series expansion of $\mathrm{e}^{m(t+t^2+t^3)}$, determine an expression in terms of m for $\mathrm{P}(Y = 4)$. [9]

15. [Maximum mark: 9]

An ellipse E has equation $x^2 + 2y^2 = 2$. The point P has coordinates (x_1, y_1) and is external to the ellipse.

(a) Write down the equation of the line L with gradient m passing through the point P. [1]

(b) Show that the x coordinates of the points of intersection of the line L and the ellipse E are given by the roots of the quadratic equation

$$x^2(1 + 2m^2) + 4mx(y_1 - mx_1) + 2y_1^2 + 2m^2x_1^2 - 4mx_1y_1 - 2 = 0.$$ [3]

(c) Show that the condition for the line L to be a tangent to E is given by

$$m^2(x_1^2 - 2) - 2mx_1y_1 + y_1^2 - 1 = 0.$$ [3]

(d) Hence show that the equation of the locus of points from which the two tangents to E are perpendicular is $x^2 + y^2 = 3$. [2]